A Possible Gauge Formulation for Gravity?

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Abstract

A possible Yang-Mills like lagrangian formulation for gravity is explored. The starting point consists on two next assumptions. First, the metric is assumed as a real map from a given gauge group. Second, a gauge invariant lagrangian density is considered with the condition that it is related to the Einstein one up to a bound term. We study a stationary solution of the abelian case for the spherical symmetry, which is connected to the Möller's Maxwell like formulation for gravity. Finally, it is showed the consistence of this formulation with the Newtonian limit.

1 Introduction

From appearance of General Relativity theory, Quadratic Lagrangian Formulations (QLF) for gravity in terms of Riemann-Christofell tensor have been considered [1, 2, 3, 4, 5, 6, 7, 8, 9]. Sometimes these models are called "Yang-Mills type formulations". In QLF, Einstein field equations are recovered from Palatini's variational principle [8]. On the other hand, the addition of terms which are quadratic in curvature to the Einstein gravitational Lagrangian yields a theory where the renormalization problems are much less severe [10]. This is similar to the situation in the Yang-Mills theories which are renormalizable [11, 12]. These results had motivated some authors to propose, for the gravitational lagrangian, expressions which are quadratic in curvature [13, 14, 15, 16].

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In a parallel way to this current research, and immediately after the formulation of Yang-Mills theory (YMT) [17], R. Utiyama [18] was the first to recognize the gauge "character" of gravitational field. Taking into account the protagonical role of YMT in the electro-weak model [19, 20, 21], one could think that this kind of theory is a candidate for description and quantization of fundamental interactions. Thus, there is a significant reason to considerate gauge formulations for gravity. Here, we are referring to constructions where gravitation is described by means of agauge connection on a certain fibre bundle. Within the variety of these theories [22, 23, 24, 25, 26, 27, 28], it must be detached the Hamiltonian formulation of A. Ashtekar [25] from which a program for quantization of gravity could be performed.

In this paper we present a possible quadratic gauge lagrangian formulation (i.e., Yang-Mills like construction) for gravity linear in Ricci curvature, following the YMT and in contrast to QLF. The starting point consists to consider real mappings of a connection in order to obtain a metric. This idea is not new, i.e., in [28] are presented new gauge theories of conformal space-time symmetries which merge features of YM theory and gravity.

The paper is organized as follows. In section 2, we present two primary considerations for the construction of a gauge invariant Lagrangian formulation, starting from the existence of a one-form connection that transforms under a given Lie group G. In section 3, field equations and the relationship with the Einstein ones are presented. Next, in section 4, we study the abelian theory computing a stationary solution for the connection. It is shown that there is a particular solution proportional to $U(1)$ connection of Maxwell type formulation of Möller for gravity. Finally, in section 5 the consistence between the Yang-Mills like field equations for gravity and Newtonian limit is discussed. We conclude with some remarks.

2 A gauge Lagrangian formulation

Let G be a Lie group, simple and arbitrary with generators t_a, $a = 1, 2, ..., N$. Let M^4 be a differentiable but not necessarilly contractible base manifold. Thus, a fibre bundle with G as structure group (principal fibre bundle) can be defined, assuming that the fibre proyection, the transition functions, etc. are given. A one-form connection arise with components $A_\mu = A_\mu^a t_a (\mu = 0, 1, 2, 3)$, and transforms under the gauge group in a usual manner

$$A'_\mu = U A_\mu U^{-1} + U \partial_\mu U^{-1} , \; U \in G. \tag{1}$$

Next, gauge invariant maps from G into reals numbers are considered. There are many ways to construct such maps. For example, we can perform gauge invariant combinations on functional traces in powers of $\Lambda_\mu(x) \equiv A_\mu(x) - \bar{A}_\mu(x)$, where $\bar{A}_\mu$ is the background connection [29] (arbitrary element of the fibre). We will call this type of maps a Local Maps (LM).

On the other hand, we can considerate Non Local Maps (NLM) that involve the Wilson functional or "Wilson Loop", which is defined as the holonomy trace: $W(c) = tr H(c) \equiv tr P \exp\left(i \oint_c A\right)$, with c an element of the Group of Loops [30] inscribed in 3+1 space-time.

Given the above definitions we present two primary considerations which will constitute a possible gauge formulation for gravity:

1.- Assuming the existence of maps from G onto reals numbers $R1$, the metric tensor in 3+1 dimension is realized with functionals that are gauge invariant under eq.(1) :

$$L.M. : \; g_{\mu\nu}(x) \equiv S_{\mu\nu}\left(A(x) - \bar{A}(x)\right) \in R1 , \tag{2}$$

or

$$N.L.M. : \; g_{\mu\nu}(x) \equiv S_{\mu\nu}\left(A(x) - \bar{A}(x), W(c)\right) \in R1 , \tag{3}$$

For any type of formulation, these relations will be written in the form: $g_{\mu\nu}(x) \equiv S_{\mu\nu}(A(x))$, where $S_{\mu\nu}$ is some functional of $A(x)$.

2.- The lagrangian density is

$$L_G = -\frac{1}{4k}\sqrt{-S(A)}F_{\mu\nu}^a(A)F^{a\mu\nu}(A) ,\tag{4}$$

where $F_{\mu\nu}^a(A) = A_{\nu;\mu}^a - A_{\mu;\nu}^a + C^{abc}A_\mu^b A_\nu^c$ is the Yang-Mills curvature, C^{abc} the completely antisymmetric structure constant and k a real constant. We will assume that eq.(4) is a Lagrangian reformulation for Einstein theory, if the action $I_G = \int d^4x L_G\left(A_\mu(x)\right)$ is equal to the Einstein one, $I_R = 1/16\pi \int d^4x \sqrt{-g(x)}R\left(g_{\mu\nu}(x)\right)$ for a given gauge group G. This is,

$$I_G = I_R .\tag{5}$$

In virtue of the arbitrariness integration region we can say that:

$$-\frac{1}{4k}F_{\mu\nu}^a(A(x))F^{a\mu\nu}(A(x)) = \frac{1}{16\pi}R\left(S_{\mu\nu}\left(A(x)\right)\right) + \Omega_{;\mu}^\mu(x) ,\tag{6}$$

where $\Omega^\mu(x)$ is a function of order r^{-n} with $n > 2$, for $r \longrightarrow \infty$. This condition on $\Omega^\mu(x)$ establishes that the constraint (6) (or "non-holonomical" constraint, due to arbitrariness of $\Omega^\mu(x)$) selects the physical field $A(x)$ for a given functionals $S_{\mu\nu}$.

These considerations can be argued as follows. The first one presents the connection A_μ as the fundamental field on the way to enlarge the symmetries, yielding an invariant formulation under General Transformation of Coordinates and gauge transformations. Further, the only propagated degrees of freedom in this formulation corresponds to the field A_μ. Thus, the number of local degree of freedom can be fixed taking an adecuate gauge group G.

The discussion about which group G can be choosen, in order to match degrees of freedom between this gauge formulation and gravity, is still open. The complete constraint system will be considered elsewhere.

The second consideration is connected with the dynamical aspect of the gauge field. A simplest gauge invariant first order lagrangian density is proposed. At the same time, a non-holonomical constraint on field A_μ is presented.

When a particular form of $S_{\mu\nu}$ is performed (i.e., combinations in powers of traces on $\Lambda_\mu \equiv A_\mu - \bar{A}_\mu$, etc.), the Yang-Mills like equations (see eq.(10)) are been prepared to conduce a certain solution, but this $S_{\mu\nu}$ must be consistent with constraint (6).

3   Dynamics

Here, a variational analysis on the total action $I = I_G + I_M$ is performed. I_M is the matter fields action with variation $\delta I_M = -1/2 \int d^4x \sqrt{-g} T_M^{\alpha\beta} \delta g_{\alpha\beta}$. Then, thinking in matter fields as external ones, arbitrary variations on A_μ yields

$$0 = \int d^4x \sqrt{-S} \left[-\frac{1}{2} \left(T_M^{\alpha\beta} + T_F^{\alpha\beta} \right) \delta S_{\alpha\beta}(A) + \frac{1}{k} (D_\mu F^{\mu\lambda})^b \delta A_\lambda^b \right], \tag{7}$$

where $T_M^{\alpha\beta}$ is the matter stress tensor, $T_F^{\alpha\beta}$ is the Yang-Mills stress tensor defined by:

$$T_F^{\alpha\beta} = -\frac{1}{k} \left(F_\sigma^{a\alpha} F^{a\sigma\beta} - \frac{S^{\alpha\beta}}{4} F_{\mu\nu}^a F^{a\mu\nu} \right), \tag{8}$$

where $S^{\alpha\beta}$ satisfies $S^{\alpha\beta} S_{\alpha\mu} = \delta_\mu^\beta$ and D_μ is a covariant derivative under gauge transformations and Coordinates transformations defined by

$$(D_\mu F^{\mu\nu})^b \equiv F_{;\mu}^{b\mu\lambda} + C^{bac} A_\mu^a F^{c\mu\lambda} . \tag{9}$$

Here, $F_{;\mu}^{b\mu\lambda} = \frac{1}{\sqrt{-S}} \partial_\mu \left(\sqrt{-S} F^{b\mu\lambda} \right)$.

With an arbitrary δA_μ, for any formulation (LM or NLM), the field equations are:

$$(D_\mu F^{\mu\lambda})^b = \frac{k}{2} \left(T_M^{\alpha\beta} + T_F^{\alpha\beta} \right) M_{b\alpha\beta}^\lambda , \tag{10}$$

where the object

$$M_{b\alpha\beta}^\lambda \equiv \frac{\partial S_{\alpha\beta}(A)}{\partial A_\lambda^b} , \tag{11}$$

represents the "Jacobian" of the map that goes from G to $R1$. In general it can be observed that the solutions of eq.(10) will depend on which prescription we take for the functional $S_{\mu\nu}$ (satisfaying the constraint (6)).

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In order to close this section we want to comment about the relationship between the Yang-Mills like equations and Einstein ones for any gauge group G. The gravity equations are:

$$R^{\alpha\beta} - \frac{g^{\alpha\beta}}{2}R + 8\pi T_M^{\alpha\beta} = 0 \; , \tag{12}$$

with $R^{\alpha\beta}$ the Ricci tensor. Let us call $N^{\alpha\beta} = R^{\alpha\beta} - \frac{g^{\alpha\beta}}{2}R + 8\pi T_M^{\alpha\beta}$ a non null object, when $R^{\alpha\beta}$ is not valued on Einstein equations (eq.(12)). Then, equalizing the total action $(I = I_G + I_M)$ with the obtained one from the Einstein theory $(I = I_R + I_M)$, and taking into account an arbitrary variation on fields A_λ^b with $\delta S_{\alpha\beta} = M_{b\,\alpha\beta}^\lambda \delta A_\lambda^b$, the following equations can be obtained:

$$\left(D_\mu F^{\mu\lambda}\right)^b - \frac{k}{2}\left(T_M^{\alpha\beta} + T_F^{\alpha\beta}\right)M_{b\,\alpha\beta}^\lambda = -\frac{k}{16\pi}N^{\alpha\beta}M_{b\ \alpha\beta}^\lambda \; . \tag{13}$$

If eq.(12) (in other words, $N^{\alpha\beta} = 0$) is introduced in eq.(13), the dynamical equations (10) arise. However, in general would be possible to find solutions for Yang-Mills like equations for a given functional $S_{\mu\nu}(A(x))$ does not corresponds to general relativity solutions (i.e., $S_{\mu\nu}(A(x)) \neq g_{\mu\nu}(x)$). This means in general that the space of solutions of Yang-Mills like equations would contains the Einstein's one.

4 Non trivial vacuum solution for the abelian theory

In order to obtain vacuum $(T_M^{\alpha\beta} = 0)$ stationary solutions for the abelian case we take $G = U(1) \times \overset{N}{...} \times U(1)$. In other words, we have N generators that satisfy a Lie Algebra with $C^{abc} = 0$ for all $a, b, c = 1, 2, ...N$. Moreover, the physical system consists in a compact spherical symmetric stationary object. This fact allows to assume a stationary connection of the type $A_\mu^a = A_\mu^a(r)$.

Thus, out of the compact object, the field equations looks as:

$$F_{;\mu}^{b\mu\lambda} = \frac{k}{2}T_F^{\alpha\beta}M_{b\,\alpha\beta}^\lambda \; . \tag{14}$$

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If one want to solve these equations, we can still give a special form to the connection and metric. Thinking in $A_\mu^a(r)$, we take the electrostatic ansatz:

$$A_0^a \neq 0 , \tag{15}$$

$$A_k^a = 0 . \tag{16}$$

For the metric we choose a Schwarzschild form (diagonal and non time dependent):

$$diag(g_{00}\left(A_0^b\right), -1/g_{00}\left(A_0^b\right), r2, r^2 sen2\theta), \tag{17}$$

that will give an ansatz for the form of functional $S_{\mu\nu}(A(x))$.

Equations (15), (16) and (17) in (14) give

$$\frac{d^2 A_0^a}{dr2} + \frac{2}{r}\frac{dA_0^a}{dr} = 0 , \tag{18}$$

$$A_k^a = 0, \tag{19}$$

and the solution is

$$A_0^a = -n^a + \frac{b^a}{r} , \tag{20}$$

$$A_k^a = 0 , \tag{21}$$

with constants n^a and b^a. From eq.(6) is easy to probe that $\Omega^1(x) \sim O(1/r^3)$. This shows that the Schwarzschild ansatz (see eq.(17)) is satisfactory.

An interesting case of eqs.(20) and (21) occurs if we put $b^a = n^a 2m$, with the Schwarzschild radius $2m$:

$$A_\mu^a = n^a g_{0\mu} , \tag{22}$$

pointing that A_μ^a is proportional to the gravitational four-potential of the Maxwell like formulation that C. Möller [31] introduced in the gravitational energy localization problem. In that reference, the author defines a $U(1)$ potential:

$$A_\mu = g_{0\mu} \ , \tag{23}$$

covariant under the Space-Time Orthogonal transformations subgroup, given by:

$$x'^i = f^i(x^j) \ , \tag{24}$$

$$x'^0 = x0. \tag{25}$$

Thus, relation (22) have the same subgroup of coordinate transformations.

Before ending this section, we want to explore the relation between the solution (22) and the one obtained from other types of symmetries. Is easy to probe for the Reissner-Nördstrom problem that eq.(22) solves eq.(18) up an order r^{-4} term. Obviously this is not a vacuum problem because in this symmetry there is an electrostatic charge. However, if $T_M^{\alpha\beta}$ is the stress tensor associated to a rest charge, we have

$$0 = \left(T_M^{\alpha\beta} + T_F^{\alpha\beta}\right) M_{b\,\alpha\beta}^\lambda \ , \tag{26}$$

under the ansatz (15), (16) and (17). Thus, eq.(26) in (29) throw an equation system equivalent to eqs.(18) and (19).

On the other hand, if one explore a non spherical symmetry (i.e., Kerr metric) it can be shown that eq.(22) satisfies eq.(14) up to a term of order r^{-5} (with an ansatz similar to (15) but with $A_1^a = A_2^a = 0$ and $A_3^a = A_3^a(r, \theta)$).

All this says that the stationary solutions of the abelian formulation ($G = U(1) \times \overset{N.}{...} \times U(1)$) for Reissner-Nördstrom and Kerr symmetries, can be approached to eq.(22) at the $r \to \infty$ limit. This asymptotic behaviour is just the property that C. Möller [31] needed in his formulation in order to define the total energy in a satisfactory way.

5 The Newtonian limit

A required consistency condition of this formulation is that the Newtonian limit must be recovered in the non relativistic weak field limit from the Yang-Mills like equations.

In a low speed regime ($|v^i| = |dx^i/dx0| \ll 1$) and a weak gravitational field, we take a Galilean Coordinate System x^μ with a stationary metric $g_{\mu\nu}$ that defers up to a weak perturbation ($|h_{\mu\nu}| \ll 1$) from the Minkowski metric ($\eta_{\mu\nu}$), in other words:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ , \tag{27}$$

with $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$. Next is assumed a perfect fluid in which pressure and speed are neglected in the Galilean System. Only the T_M^{00} component of material stress tensor (related with the mass density) will be considered.

In order to complete the passage to the Newtonian limit from the Yang-Mills like formulation we need to assume the next considerations. First, it is required the linear behaviour of the gravitational field. In this sense, the gauge group must be $G = U(1) \times \overset{N}{\ldots} \times U(1)$.

Second, in the weak field regime, we can think that relation (27) arises from a perturbation on the connection via eq.(2) or (3). In others words, performing an infinitesimal variation $\delta A_\mu^b(x)$ around a fixed $A_{o\mu}^b$ ($\Lambda_{o\mu} \equiv A_{o\mu} - \bar{A}_{o\mu}$ fixed too) with $S_{\mu\nu}(A_o) = \eta_{\mu\nu}$, we have

$$g_{\mu\nu}(x) = S_{\mu\nu}(A(x)) = \eta_{\mu\nu} + M_{b\mu\nu}^\lambda(A_o)\delta A_\lambda^b(x) \ . \tag{28}$$

So then, with $M_{b\mu\nu}^\lambda(A_o)$ bounded, eq.(27) can be matched with eq.(28). Moreover, the infinitesimal functions $h_{\mu\nu}(x)$ and $\delta A_\mu^b(x)$ have the same order.

On the other hand, $G = U(1) \times \overset{N}{\ldots} \times U(1)$ in eq.(10) gives

$$F_{;\mu}^{b\mu\lambda} = \frac{k}{2}\left(T_M^{\alpha\beta} + T_F^{\alpha\beta}\right)M_{b\,\alpha\beta}^\lambda \ , \tag{29}$$

and taking only first order contributions in $h_{\mu\nu}$ and $\delta A_\mu^b(x)$, the time component of the left hand side of eq.(29) yields

$$F_{;i}^{bi0} = -F_{i0,i}^b = -\nabla^2 A_0^b \ . \tag{30}$$

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The Yang-Mills stress tensor $\left(T_F^{\alpha\beta}\right)$ constitutes a quadratic order contribution in $\delta A_\mu^b(x)$. With this, the time component of the right hand side of eq.(29) is

$$\frac{k}{2}\left(T_M^{\alpha\beta}+T_F^{\alpha\beta}\right)M_{b\,\alpha\beta}^0 = \frac{k}{2}M_{b00}^0(A_o)T_M^{00} \ . \tag{31}$$

Joining eqs.(30) and (31) in (29) for $\lambda = 0$, the next N equations are obtained

$$\nabla^2 A_0^b = \alpha^b T_M^{00} \ , \tag{32}$$

where $\alpha^b = -\frac{k}{2}M_{b00}^0(A_o)$. Expression (32) is the Laplace equation for the Newtonian limit of the Yang-Mills like ones.

6   Concluding remarks

In this work, an initial idea about a possible scheme for a reformulation of gravity theory in a similar way of a Yang-Mills theory at a classical level was presented. This have been made thinking in two factibles gauge invariant types of lagrangian formulations (LM or NLM) which lead to consistent dynamical equations with the Newtonian limit.

A spherical symmetric stationary solution for the abelian case was obtained, showing the relation with the Maxwell like four-potential of Möller. It would be interesting to explore non abelian solutions for stationary spherical symmetry where a Bartnik-McKinnon [32] type ansatz can be used.

The fundamental problem oriented to the quantisation bussines and related to the Dirac's canonical analysis of constraints [33] will be considered elsewhere.

Acknowledgments

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